1 [1] $\overrightarrow{OP} = t\vec{a} + (1-t)(1-x)\vec{b}$

$s\frac{x}{2}\vec{a} + (1-s)\vec{b}$

$t = s\frac{x}{2},\ (1-t)(1-x) = 1-s$

$(1-s\frac{x}{2})(1-x) = 1-s$

$\frac{s}{2}x^2 - \frac{s}{2}x - x + \cancel{1} = \cancel{1} - s$

$(x^2-x+2)s = 2x$

$s = \frac{2x}{x^2-x+2}$

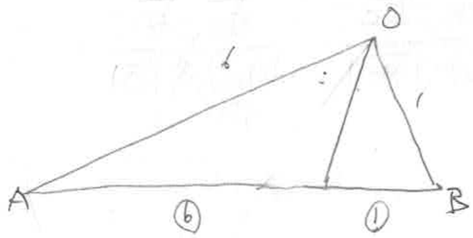
$\overrightarrow{OP} = \frac{x^2}{(x^2-x+2)}\vec{a} + \frac{x^2-3x+2}{x^2-x+2}\vec{b}$　　$-\vec{a}+\vec{b}$

[2]　$x^2 = x^2-3x+2$　$3x = 2$　$\therefore x = \frac{2}{3}$

このとき $\overrightarrow{OP} = \frac{1}{4}\vec{a} + \frac{1}{4}\vec{b} = \frac{1}{2}\left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}\overrightarrow{OM}$

$\therefore OP : PM = 1 : 1$

[3] 角の二等分線の定理より



$6x^2 = x^2 - 3x + 2$

$5x^2 + 3x - 2 = 0$

$(5x-2)(x+1) = 0$

$x = \frac{2}{5}, -1$　　$\beta = \frac{2}{5}$

[4]　$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\angle AOB = -\frac{1}{2}$

$\overrightarrow{OP}\cdot\overrightarrow{AB} = \frac{-x^2\cdot 4 + x^2\cdot(-\frac{1}{2}) + (x^2-3x+2)(-\frac{1}{2}) + (x^2-3x+2)}{x^2-x+2}$

分子 $= -3x^2 - \frac{9}{2}x + 3 = 2x^2 + 3x - 2 = 0$

$= (2x-1)(x+2)$　　$x = \frac{1}{2}, -2$

$\therefore \beta = \frac{1}{2}$

2 [1] $a_{n+1}-1 = \frac{1}{2}(a_n - 1)$

$$a_n = \frac{6}{2^{n-1}} + 1 = \frac{3}{2^{n-2}} + 1$$

[2] $\int_0^1 f_{n+1}(x)dx = \int_0^1 a_{n+1}x\,dx + \frac{1}{2}\int_0^1\left(\int_0^1 f_n(t)dt\right)dx$

$$= a_{n+1}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\int_0^1 f_n(t)dt = \frac{1}{2}a_{n+1} + \frac{1}{2}\int_0^1 f_n(t)dt$$

$$b_{n+1} = 2^{n+1}\int_0^1 f_{n+1}(t)dt = 2^n a_{n+1} + 2^n\int_0^1 f_n(t)dt$$

$$= 2^n a_{n+1} + b_n = 6 + 2^n + b_n$$

$$b_1 = 7$$

$$\therefore\ b_n = 7 + \sum_{k=1}^{n-1}(2^k + 6) = 2^n + 6n - 1$$

[3]

$f_1(x) = 7x$, $f_1(x) = 0$ のとき $x = 0$

$n \geqq 2$のとき

$$f_n(x) = a_n x + \frac{1}{2}\int_0^1 f_{n-1}(t)dt$$

$$= \left(\frac{3}{2^{n-2}} + 1\right)x + \frac{b_{n-1}}{2^n}$$

$$= \left(\frac{3}{2^{n-2}} + 1\right)x + 1 + \frac{6n-7}{2^n}$$

$f_n(x) = 0$のとき

$$x = -\frac{2^n + 6n - 7}{2^n + 12}$$ $n=1$ のときも成立

3 (1) $f(x) = x + (4x^2+1)^{\frac{1}{2}}$

$f'(x) = 1 + \frac{1}{2}(4x^2+1)^{-\frac{1}{2}} 8x$

$= 1 + \frac{4x}{\sqrt{4x^2+1}}$

$f'(x)=0$ のとき $\frac{4x}{\sqrt{4x^2+1}} = -1$

$-4x = \sqrt{4x^2+1} \quad (x<0)$

$16x^2 = 4x^2+1$

$x = -\frac{1}{2\sqrt{3}} = -\frac{\sqrt{3}}{6}$

x		$-\frac{1}{2\sqrt{3}}$	
f'	$-$	0	$+$
f	↘	$\frac{\sqrt{3}}{2}$	↗

$f(-\frac{1}{2\sqrt{3}}) = -\frac{1}{2\sqrt{3}} + \sqrt{\frac{4}{12}+1} = -\frac{1}{2\sqrt{3}} + \frac{4}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$

(2) (1)

$y = \left(1 + \frac{4p}{\sqrt{4p^2+1}}\right)(x-p) + p + \sqrt{4p^2+1}$

$= \left(1 + \frac{4p}{\sqrt{4p^2+1}}\right)x + \frac{1}{\sqrt{4p^2+1}}$

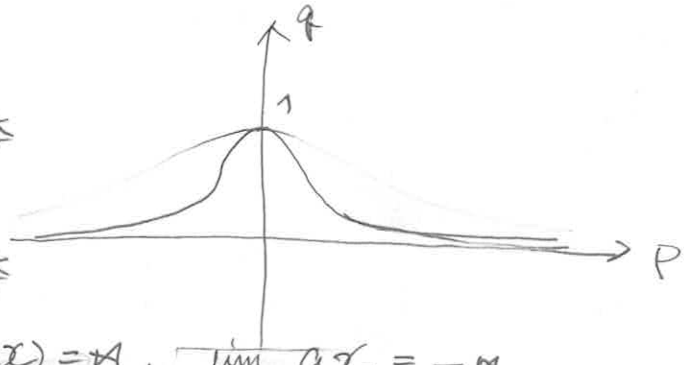
(2)

$q = \frac{1}{\sqrt{4p^2+1}}$

$q \leqq 0$, $1<q$ のとき 0本

$q=1$ のとき 1本

$0<q<1$ のとき 2本



(3) $a<0$ のとき $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to\infty} ax = -\infty$

$a \geqq 0$ のとき

$x + \sqrt{4x^2+1} - ax = (1-a)x + \sqrt{4x^2+1}$

$= \frac{(1-a)^2x^2 - (4x^2+1)}{(1-a)x - \sqrt{4x^2+1}} = \frac{(a^2-2a-3)x^2 - 1}{(1-a)x + \sqrt{4x^2+1}} = \frac{(a-3)(a+1)x - \frac{1}{x}}{(1-a) - \sqrt{4+\frac{1}{x^2}}}$

$\therefore a=3$

このとき、 $\lim_{x\to\infty} f(x) - ax = \frac{-\frac{1}{x}}{-2-\sqrt{4+\frac{1}{x^2}}} = \frac{0}{-2-\sqrt{4+0}} = 0$

4 [1] (1)

$$\int \sin x \sin 2x\,dx = \int \sin x \cdot 2\sin x\cos x\,dx$$

$$= 2\int \sin^2 x\cos x\,dx = 2\int t^2 \frac{dt}{d\cancel{x}}\,d\cancel{x}$$

$(t = \sin x)$

$$= \frac{2}{3}t^3 + C = \frac{2}{3}\sin^3 x + C$$

$(fg)' = f'g + fg'$

(2)

$$\int x\sin nx\,dx = \int x\left(-\frac{\cos nx}{n}\right)' dx$$

$$= -\frac{x\cos nx}{n} + \int \frac{\cos nx}{n}\,dx + C$$

$$= -\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} + C$$

[2] $(x + a\sin x + b\sin 2x)^2 = f(x)$

$$= x^2 + a^2\sin^2 x + b^2\sin^2 2x + 2ax\sin x + 2bx\sin^2 2x + 2ab\sin x\sin 2x$$

$$\int_0^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^3}{3}$$

$$\int_0^{\pi} a^2\sin^2 x\,dx = a^2\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx = \frac{a^2}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{a^2}{2}\pi$$

$$\int_0^{\pi} b^2\sin^2 2x\,dx = b^2\int_0^{\pi}\frac{1-\cos 4x}{2}\,dx = \frac{b^2}{2}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi} = \frac{b^2}{2}\pi$$

$$\int_0^{\pi} 2ax\sin x\,dx = 2a\left[-x\cos x + \sin x\right]_0^{\pi} = 2\pi a$$

$$\int_0^{\pi} 2bx\sin 2x\,dx = 2b\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi} = -\pi b$$

$$\int_0^{\pi} 2ab\sin x\sin 2x\,dx = 2ab\left[\frac{2}{3}\sin^3 x\right]_0^{\pi} = 0$$

$$\int_0^{\pi} f(x)\,dx = \frac{\pi}{2}a^2 + 2\pi a + \frac{\pi}{2}b^2 - \pi b + \frac{\pi^3}{3}$$

$$= \frac{\pi}{2}(a+2)^2 + \frac{\pi}{2}(b-1)^2 - \frac{5}{2}\pi + \frac{\pi^3}{3}$$

$a=-2,\ b=1$ のとき 最小値 $-\frac{5}{2}\pi + \frac{\pi^3}{3}$